

Comparison between solutions of Touchard and Gegenbauer polynomials for integro-differential equations

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Abstract: In this work, we applied the polynomial method of the Touchard and Gegenbauer type with the Galerkin method to a certain class of integro-differential equations of the Volterra type of the second type [5, 7] and of the Fredholm type of the second type, where we also touched upon integro-differential equations of the second order. Second, although Touchard polynomials are not orthogonal while Gegenbauer polynomials are orthogonal, the results obtained were different depending on the use of each Galerkin-Touchard or Galerkin-Gegenbauer polynomial. However, we found that the method is effective and the results are good through the examples presented.

Keywords: Integro-differential equations, Touchard polynomials, Gegenbauer polynomials, Galerkin method, Approximate Solution.

Introduction

Integral and Integro-differential equations are considered one of the most important equations in pure and applied mathematics. Integro-differential equations have a very important role in cutting edge science and technology such as heat transfer, physics, mechanics and biological applications. Many references can be found that talk about Integro-differential equations see [3, 6, 12].

Many researchers have developed numerical methods to solve this type of equations [2, 11, 13, 19]. In this research, we used an approximate method to solve the Fredholm and Volterra Integro-differential equation of the second type. This method is a method based on applying the Galerkin method with different polynomials of the Touchard type, which always gives good results; see [1, 2, 14]. Because of its good properties, especially the derivation, where when we derive Touchard's polynomials, we find that they have a relationship to the original Touchard polynomials, meaning that. $T' = nT$. This property makes it very suitable for obtaining good results in approximate solutions of Integro-differential equations, on the other hand. Applying Gegenbauer polynomials, which are considered somewhat complicated considering that they are orthogonal $\langle G_i, G_j \rangle = 0$ only for every, $j = i + 1$ see [8]. However, we obtain excellent approximate solutions. Numerical examples and results were presented in the fifth paragraph, where we presented several examples where we obtained good results, knowing that whenever the higher the degree of polynomials, the better the results and the more effective the method.

P-Galerkin Method

We consider the following integro-differential equation

$$u'(x) = f(x) + \int_{a(x)}^{b(x)} k(x, t) u(t) dt \quad (1)$$

$$u(a) = \alpha. \quad (2)$$

Integro-differential equations can also be defined in the following general form

$$\sum_{i=0}^n g u^{(i)}(x) = f(x) + \int_{a(x)}^{b(x)} k(x, t) u(t) dt \quad u^{(i)}(a) = \alpha_i.$$

Such that $u(x)$ is the unknown function, $k(x, t)$ is the known continuous function, $f(x)$ is a given function, g is a constant parameter.

The method uses P – Galerkin polynomials (P It represents the Touchard or Gegenbauer polynomials), to approximate the solution in a closed and finite interval

$$u(x) \simeq u_n(x) = \sum_{i=0}^n a_i P_i(x), a \leq x \leq b, \quad (3)$$

Where $P(x)$, is the Touchard or Gegenbauer polynomial constructed and valid in the interval $[a, b]$ Thus, by differentiating (3) with respect to x , we have

$$u'(x) \simeq u'_n(x) = \sum_{i=0}^n a_i P'_i(x) \quad (4)$$

By replacing (3) and (4) in (1), we obtain

$$\sum_{i=0}^n a_i P'_i(x) = f(x) + \int_a^x k(x, t) \sum_{i=0}^n a_i P_i(t) dt \quad (5)$$

We determined the unknown coefficients a_i using the P-Galerkin idea by multiplying both sides of (5) by $P_j(x)$ and then integrating with respect to x from a to b . Thus, we obtain

$$\begin{aligned} & \sum_{i=0}^n a_i \int_a^b P'_i(x) P_j(x) dx \\ &= \int_a^b P_j(x) f(x) dx + \int_a^b \int_a^x k(x, t) \sum_{i=0}^n a_i P_i(t) P_j(x) dt dx, j = 0, 1, \dots, n \end{aligned} \quad (6)$$

This process generates a system of linear equations for the unknown $a_i, i = 0, 1, \dots, n$ as well as the conditions

$$\sum_{i=0}^n a_i P_i(a) = \alpha. \quad (7)$$

For the same number of equations in the linear system. The unknown parameters are determined by solving systems (6) and (7). The values of the constants obtained are then substituted into (3) to obtain the approximate solution.

Touchard Polynomials

Touchard polynomials, studied by J. Touchard, composed of (a polynomial sequence of binomial type). The touchard polynomials [9, 10, 18] are given as

$$T_n(x) = \sum_{k=0}^n s(n, k) x^k = \sum_{k=0}^n \binom{n}{k} x^k,$$

Where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, n is the degree of polynomials, k is the index of polynomials and, x is the variable.

The first Touchard polynomials on the interval $[0,1]$ for $n = 3$ are

$T_0(x) = 1$	$T_1(x) = 1 + x$
$T_2(x) = 1 + 2x + x^2$	$T_3(x) = 1 + 3x + 3x^2 + x^3$

Derivatives of Touchard Polynomials

Are

$$\frac{d}{dx} T_n(x) = \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=1}^n \binom{n}{k-1} x^{k-1}$$

Gegenbauer Polynomials

The $n^{\text{ième}}$ Gegenbauer polynomials $G_n(x)$ is defined by $G_0(x) = 1$ and the following recursion [8].

$$G_n(\alpha, x) = \frac{\Gamma(2\alpha + n) \cdot \Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(2\alpha) \Gamma\left(\alpha + n + \frac{1}{2}\right)} \cdot J\left(n, \alpha - \frac{1}{2}, \alpha - \frac{1}{2}, x\right).$$

The Gegenbauer polynomials G_n are polynomials with rational coefficients

$G_0 = 1$	$G_1 = 2x$
$G_2 = 2x^2 - 1$	$G_3 = \frac{8}{3}x^3 - 2x$

Numerical Results

In this section, we intend to show the efficiency of the P-Galerkin method for solving Fredholm and Volterra integro-differential equations of the second kind by Touchard and Gegenbauer polynomials by presenting six illustrative examples. The absolute error for this formulation is defined by

$$E(x) = |u(x) - u_n(x)|.$$

Example 1. consider Volterra's linear integro-differential equation.

$$u'(x) = -1 - \frac{1}{2}x^2 - xe^x \int_0^x tu(t)dt,$$

With the initial condition $u(0) = 0$.

The exact solution is given by

$$u(x) = 1 - \exp(x).$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 1. We present the exact and approximate solutions of the equation in Example 1 in some arbitrary points, the error for N=10.

x	<i>Ex. sol u</i>	<i>App. sol, T_n</i>	<i>Error G_n</i>	<i>Error T_n</i>
0.0	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	-0.1052	-0.1052	1.181224e-07	4.018680e-10
0.2	-0.2214	-0.2214	1.240865e-07	2.001042e-10
0.3	-0.3499	-0.3499	4.428027e-08	2.824012e-10
0.4	-0.4918	-0.4918	8.055553e-08	3.386343e-10
0.5	-0.6487	-0.6487	9.539796e-08	1.475158e-10
0.6	-0.8221	-0.8221	7.641807e-09	3.267855e-10
0.7	-1.0138	-1.0138	7.999178e-08	5.701661e-12
0.8	-1.2255	-1.2255	6.585608e-08	2.418297e-10
0.9	-1.4596	-1.4596	3.825747e-08	2.088929e-10
1	-1.7183	-1.7183	2.750349e-10	3.486100e-14

Example 2. consider the linear integro-differential equation of Volterra.

$$u'(x) = 2 - \frac{x^2}{4} + \frac{1}{4} \int_0^x u(t) dt,$$

With the initial condition $u(0) = 0$.

The exact solution is given by

$$u(x) = 2x.$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 2. We present the exact and approximate solutions of the equation in Example 2 in some arbitrary points, the error for N=10.

x	<i>Ex. sol u</i>	<i>App. sol, G_n</i>	<i>Error G_n</i>	<i>Error T_n</i>
0.0	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	0.2000	0.2000	5.551115e-17	4.546100e-11
0.2	0.4000	0.4000	5.551115e-17	3.050771e-11
0.3	0.6000	0.6000	1.110223e-16	2.620604e-11
0.4	0.8000	0.8000	3.330669e-16	4.714906e-11
0.5	1.0000	1.0000	2.220446e-16	9.988899e-12
0.6	1.2000	1.2000	2.220446e-16	4.605494e-11
0.7	1.4000	1.4000	2.220446e-16	6.851408e-12
0.8	1.6000	1.6000	2.220446e-16	3.263234e-11
0.9	1.8000	1.8000	2.220446e-16	3.237899e-11
1	2.0000	2.0000	4.440892e-16	8.881784e-16

Example 3. consider Fredholm's linear integro-differential equation

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t) dt,$$

With the initial condition $u(0) = 0$.

The exact solution is given by

$$u(x) = xe^x$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 3. We present the exact and approximate solutions of the equation in Example 3 in some arbitrary points, the error for $N=10$.

x	<i>Ex. sol u</i>	<i>App. sol, G_n</i>	<i>Error G_n</i>	<i>Error T_n</i>
0.0	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	0.1105	0.1105	2.510347e-07	1.684015e-10
0.2	0.2443	0.2443	2.635946e-07	5.280915e-12
0.3	0.4050	0.4050	9.387603e-08	1.925111e-10
0.4	0.5967	0.5967	1.711263e-07	5.011702e-11
0.5	0.8244	0.8244	2.025128e-07	1.482066e-10
0.6	1.0933	1.0933	1.631099e-08	4.893796e-11
0.7	1.4096	1.4096	1.696869e-07	9.171108e-11
0.8	1.7804	1.7804	1.399741e-07	5.911982e-11
0.9	2.2136	2.2136	8.117024e-08	4.014122e-12
1	2.7183	2.7183	7.874119e-10	5.684342e-14

Example 4. consider Fredholm's linear integro-differential equation

$$u'(x) = 3e^{3x} - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xt u(t) dt,$$

With the initial condition $u(0) = 1$.

The exact solution is given by

$$u(x) = e^{3x}$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 4. We present the exact and approximate solutions of the equation in Example 4 in some arbitrary points, the error for $N=10$.

x	<i>Ex. sol u</i>	<i>App. sol, T_n</i>	<i>Error G_n</i>	<i>Error T_n</i>
0.0	1.0000	1.0000	0.000000e+00	0.000000e+00
0.1	1.3499	1.3499	2.224544e-06	1.821110e-07
0.2	1.8221	1.8221	2.332818e-06	2.196689e-07
0.3	2.4596	2.4596	8.192581e-07	1.080370e-07
0.4	3.3201	3.3201	1.530228e-06	1.372972e-07
0.5	4.4817	4.4817	1.795768e-06	2.037644e-07
0.6	6.0496	6.0496	1.589366e-07	3.222279e-09

0.7	8.1662	8.1662	1.517251e-06	1.723809e-07
0.8	11.0232	11.0232	1.247652e-06	1.568693e-07
0.9	14.8797	14.8797	7.209476e-07	9.678357e-08
1	20.0855	20.0855	8.744678e-09	8.384404e-13

Example 5. consider Fredholm's linear integro-differential equation

$$u'(x) = 3 + 6x + \int_0^1 xtu(t) dt$$

With the initial condition $u(0) = 0$.

The exact solution is given by

$$u(x) = 3x + 4x^2$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 5. We present the exact and approximate solutions of the equation in Example 5 in some arbitrary points, the error for $N=10$.

x	<i>Ex. sol u</i>	<i>App. sol, G_n</i>	<i>Error G_n</i>	<i>Error T_n</i>
0.0	0.00000	0.00000	0.000000e+00	0.000000e+00
0.1	0.34000	0.34000	8.883658e-07	0.000000e+00
0.2	0.76000	0.76000	9.327980e-07	0.000000e+00
0.3	1.26000	1.26000	3.322368e-07	0.000000e+00
0.4	1.84000	1.84000	6.056290e-07	0.000000e+00
0.5	2.50000	2.50000	7.165786e-07	0.000000e+00
0.6	3.24000	3.24000	5.761295e-08	0.000000e+00
0.7	4.06000	4.06000	6.006336e-07	0.000000e+00
0.8	4.96000	4.96000	4.951495e-07	0.000000e+00
0.9	5.94000	5.94000	2.874869e-07	0.000000e+00
1	7.00000	7.00000	2.488216e-09	0.000000e+00

Example 6. Consider linear Volterra integro-differential equation of second kind

$$u''(x) = x + \int_0^x (x-t)u(t) dt$$

With the initial condition $u(0) = 0, u'(0) = 1$

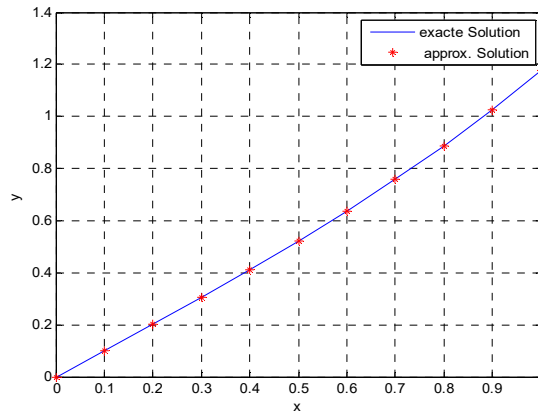
Where the function $f(x)$ is chosen such that the exact solution is given by

$$u(x) = \sinh(x)$$

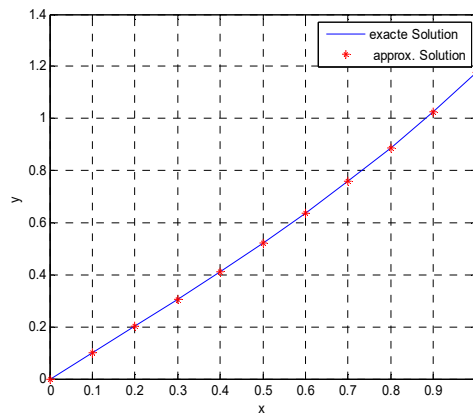
The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 6. We present the exact and approximate solutions of the equation in Example 6 in some arbitrary points, the error for $N=10$.

x	<i>Ex. sol u</i>	<i>App. sol, G_n</i>	<i>App. sol, T_n</i>	<i>ErrorG_n</i>	<i>ErrorT_n</i>
0.0	0.0000	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	0.1002	0.1002	0.1002	1.484785e-13	3.461069e-08
0.2	0.2013	0.2013	0.2013	2.196854e-13	1.709290e-07
0.3	0.3045	0.3045	0.3045	1.260103e-13	1.162768e-07
0.4	0.4108	0.4108	0.4108	3.160805e-13	1.272785e-07
0.5	0.5211	0.5211	0.5211	1.110223e-15	1.270703e-07
0.6	0.6367	0.6367	0.6367	1.942890e-14	6.612520e-08
0.7	0.7586	0.7586	0.7586	3.588241e-13	1.112242e-07
0.8	0.8881	0.8881	0.8881	3.059775e-13	8.854962e-09
0.9	1.0265	1.0265	1.0265	3.417266e-13	2.446322e-08
1	1.1752	1.1752	1.1752	5.226930e-13	3.258046e-09



Graph for example 6 using. Gegenbauer-Galerkin



Graph for example 6 using. Touchard-Galerkin

Conclusion

In this work, we presented and studied a numerical solution method for integro-differential equations based on Touchard and Gegenbauer polynomials. At the end of our work, we

illustrated programming examples using the MATLAB numerical calculation program, where we estimated method errors and compared the approximate solutions obtained using Touchard and Gegenbauer separately, while also comparing with the exact solution.

To test the effectiveness of the method and estimate the accuracy through our results. We noticed that the results were good, that is to say that the greater the degree of N , the closer the results was to precision, and the more efficient the method.

It was also pointed out here that the method can be applied to integro-differential equations of higher orders.

It was also pointed out here that the method can be applied to higher-order integro-differential equations, noting that the results were better for the Volterra type integro-differential equations when using Gegenbauer-Galerkin, and exactly the opposite when Touchard-Galerkin used the Fredholm type integro-differential equations.

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