

A Characterization of Sub-b- E-convex Functions

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Abstract: The concept of sub-b-E-convex functions is introduced as a novel category of generalized sub-b-convex functions and E-convex functions. It combines the definitions of sub-b-convex functions and E-convex functions. In addition. It is shown that a radially lower semi-continuous function $f: M \rightarrow \mathbb{R} \cup \{+\infty\}$ is sub-b-E-convex on M with respect to map $b: M \times M \times [0,1] \rightarrow \mathbb{R}$, if and only if for all $x, y \in (0, 1)$ there exist $\lambda = \lambda(x,y) \in (0, 1)$ such that

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)) + b(x, y, \lambda)$$

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Introduction

Convexity plays a significant part in numerous disciplines such as mathematics, management science, economics, engineering, and various applied sciences. It is mostly vital to the analysis of optimization problems where convexity is determined by different appropriate properties. However, so far all real-life problems can not be described by a convex mathematical model. That is why we generalized convex function because it gives more accurate results of reality in many cases. Owing to the importance of the convexity and generalized convexity in the study of optimality to solve mathematical programming, particularly in optimization theory.

Several researchers worked a lot on the generalized convex functions, E.A. Youness (1999) [1] presented a category of sets and a category of functions known as E-convex sets and convex functions, respectively. These categories were developed by modifying the definitions of convex sets and convex functions. In his study, X.M. Yang (2001) [2] provided examples to support and enhance the work of E.A. Youness [1]. To access other findings on generalized E-convex location, please consult references [6,7, 8].Currently, there is significant research interest in these types of generalized convex functions. In their 2009 paper, Yu-Ru Sayu et al. [3] established a category of functions known as E-b-vex functions. These functions are defined as a broader version of both b-vex functions and E-vex functions. In their 2012 study, M.T. Chao et al. [4] examined a novel form of generalized sub-b-convex functions and a specific category of sub-b-convex sets. They also provided the necessary conditions for optimality in both unconstrained and equality constrained sub-b convex programming. We have discovered a category of generalized E-convex functions and sub-b-convex functions that possess certain qualities similar to sub-b-convex functions and even E-convex functions. Furthermore, these generalized convex functions are more encompassing than both sub-b-convex functions and E-convex functions. The extension of convexity, such as sub-b-convexity and E-convexity, has

piqued our research interest. This paper is influenced by the research studies conducted by authors [1, 4]. The objective of our study is to introduce a novel category of generalized convex functions known as sub-b-E-convex functions, and to examine various aspects of the class of functions that adhere to sub-b-convexity. We have introduced a new category of functions known as sub-b-E-convex functions, which expands upon the concept of sub-b-convexity.

Preliminaries

In this section we reviewed the definitions of sub-b-convexity , E-convexity of function and sub-b-E-convex at the beginning of this section . The class of sub-b-convex functions is defined by M.T.Chao el al [4] , while the class of E-convex functions is defined by E.A. Youness (1999) [1] and Y.R.Syau [5].

Definition 2.1.

The function $f: M \rightarrow \mathbb{R}$ is said to be a sub-b-convex function on M with the respect to map $b: M \times M \times [0,1] \rightarrow \mathbb{R}$ if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + b(x, y, \lambda)$$

Holds for all $x, y \in M$ and $\lambda \in [0,1]$. while E.A.Youness [1] and Y.R.Syau [5] considered the class of functions which is called E-convex defined in the following way:

Definition 2.2.

Let $M \subseteq \mathbb{R}^n$ be a nonempty convex set. A function $f: M \rightarrow \mathbb{R}$ is said to be: E-convex on M if for all $x, y \in M$ and $\lambda \in [0,1]$, then

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y))$$

For each $x, y \in M$ and $\lambda \in [0,1]$. Holds for all $x, y \in M$ and $\lambda \in [0,1]$. by combining Definition 2.1 and Definition 2.2 we get a new class called sub-b-E-convex which introduce the concepts of sub-b-E-convex functions .Then we study some it is basic properties.

Definitions 2.3.

The function $f: M \rightarrow \mathbb{R}$ is said to be sub-b-E-convex function on M with respect to map $b: M \times M \times [0,1] \rightarrow \mathbb{R}$ if

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)) + b(x, y, \lambda) \quad (2.1)$$

Holds for all $x, y \in M$, $\lambda \in [0, 1]$. Furthermore, if

$$f(\lambda E(x) + (1 - \lambda)E(y)) \geq \lambda f(E(x)) + (1 - \lambda)f(E(y)) + b(x, y, \lambda) \quad (2.2)$$

Holds for all $x, y \in M$, $\lambda \in [0, 1]$, then the function f is said to be sub-b-E-concave function. If the signs of inequalities in (2.1) and (2.2) are strict then f is called strictly sub-b-E-convex and sub-b-E-concave function respectively.

Remark 2.4.

When $b(x, y, \lambda) \leq 0$, the sub-b-E-convex function is reduce to be E-convex function.

Remark 2.5.

Sub-b-E-convex function can be concave. It is easy to see that when $(x, y, \lambda) = f(\lambda E(x) + (1 - \lambda)E(y))$, if $f(E(x)) \geq 0$ then, f is a sub-b-E-convex function. In this case if f is a concave function, then f is both sub-b-E-convex and concave function.

Theorem 2.6.

Let M be sub-b-E-convex subset of real vector space, then radially lower semi-continuous function $f: M \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be sub-b-E-convex on M with respect to map $b: M \times M \times [0, 1] \rightarrow \mathbb{R}$, if and only if for all $x, y \in M$ there exist $\lambda = \lambda(x, y) \in (0, 1)$ that satisfies inequality (2.1).

The following corollary which has been conjectured by Mirostave in the case of real-valued continuous function is immediate.

Corollary 2.7.

Let $I \subseteq \mathbb{R}$ be a nonempty interval, then a lower semi-continuous functions $f: I \rightarrow \mathbb{R} \cup \{+\infty\}$ is sub-b-E-convex on I with respect to maps $b: I \times I \times [0, 1] \rightarrow \mathbb{R}$, if and only if for all $x, y \in I$ there exist $\lambda = \lambda(x, y) \in (0, 1)$, that satisfies inequality (2.1).

It is worth noting that, lower semi-continuous assumption, can not be relaxed too much since there exist anon sub-b-E-convex $f: I \rightarrow \mathbb{R} \cup \{+\infty\}$ on I with respect to maps $b: I \times I \times [0, 1] \rightarrow \mathbb{R}$, that is lower semi-continuous every where in I except at countably points and such that for all $x, y \in I$, there exists $\lambda = \lambda(x, y) \in (0, 1)$ that satisfies inequality (2.1). Indeed it sufficient to choose $I = \mathbb{R}$, and let f indicator function of the rationals, i.e. $f(E(x)) = 1$ if x is rational and $f(E(x)) = 0$ otherwise.

Proof of Theorem 2.6

We need only to prove if part, Moreover the claim is obvious if M is empty, hence let us assume that $M \neq \emptyset$. Fix, $x, y \in M$ and define the set

$$S := \{\lambda \in [0, 1]: f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)) + b(x, y, \lambda)\}$$

Notice $S \subseteq [0, 1]$ provide that $f(E(x)) < +\infty$ and $f(E(y)) < +\infty$,

Claim1. S is an nonempty compact subset of $[0, 1]$

Proof. Note that $\{0, 1\} \subseteq S \subseteq [0, 1]$, hence S is nonempty and bounded.

Let $\{\lambda_n\}$ be a sequence in S such that $\lambda_n \uparrow \lambda > 0$ define the real sequence $\{t_n\}$ by $t_n := 1 - \frac{\lambda_n}{\lambda}$ for each n and note that $t_n \downarrow 0$. Since f is radially lower semi-continuous it follows that.

$$\begin{aligned}
& f(\lambda E(x) + (1 - \lambda)E(y)) \\
& \leq \lim_{n \rightarrow \infty} \inf f(\lambda E(x) + (1 - \lambda)E(y)) + t_n (E(y) - \lambda E(x) - (1 - \lambda)E(y)) \\
& + b(x, y, \lambda) \\
& = \lim_{n \rightarrow \infty} \inf f(\lambda_n E(x) + (1 - \lambda_n)E(y)) + b(x, y, \lambda_n) \leq \lim_{n \rightarrow \infty} \lambda_n f(E(x)) \\
& + (1 - \lambda_n)f(E(y)) + b(x, y, \lambda_n) = \lambda f(E(x)) + (1 - \lambda)f(E(y)) + b(x, y, \lambda)
\end{aligned}$$

Similarly, given a sequence $\{\lambda_n\}$ in S such that $\lambda_n \downarrow \lambda < 1$, let $\{S_n\}$ be the sequence defined by $S_n := \frac{1-\lambda_n}{1-\lambda}$ for each n and not that $s_n \downarrow 0$, then

$$\begin{aligned}
& f(\lambda E(x) + (1 - \lambda)E(y)) \\
& \leq \lim_{n \rightarrow \infty} f f(\lambda E(x)) + (1 - \lambda)(E(y)) + S_n (E(x) - \lambda E(x) - (1 - \lambda)(E(y))) \\
& + b(x, y, \lambda_n) \\
& = \lim_{n \rightarrow \infty} f f(\lambda_n E(x)) + (1 - \lambda_n)(E(y)) + b(x, y, \lambda_n) \\
& \leq \lambda E(x) + (1 - \lambda)E(y) + b(x, y, \lambda)
\end{aligned}$$

Therefore S is closed, proving the claim.

Claim2. $(\lambda_1, \lambda_2) \cap S \neq \emptyset$ for all $\lambda_1, \lambda_2 \in S$ with $\lambda_1 < \lambda_2$

Proof. Fix $\lambda_1, \lambda_2 \in S$ with $\lambda_1 < \lambda_2$ (note that this is possible since $\{0, 1\} \subseteq S$) and define

$$h = \lambda_1 E(x) + (1 - \lambda_1)E(y) + b(x, y, \lambda_1), k = \lambda_2 E(x) + (1 - \lambda_2)E(y) + b(x, y, \lambda_2)$$

Hence $h, k \in M$ and by hypothesis, there exist $\lambda = \lambda(h, k) \in (0, 1)$ such that

$$f(\lambda E(h) + (1 - \lambda)E(k)) \leq \lambda f(E(h)) + (1 - \lambda)f(E(k)) + b(x, y, \lambda)$$

At this point, define $\lambda' := \lambda\lambda_1 + (1 - \lambda)\lambda_2$ and observe that

$$\begin{aligned}
& f(\lambda' E(x) + (1 - \lambda')E(y)) + b(x, y, \lambda') = f(\lambda E(h) + (1 - \lambda)(E(k))) + b(h, k, \lambda) \\
& \leq \lambda f E(h) + (1 - \lambda)f(E(x)) + b(h, k, \lambda) \\
& \leq \lambda (\lambda_1 f(E(x)) + (1 - \lambda_1)f E(y) + b(x, y, \lambda_1)) \\
& + (\lambda_2 f(E(x)) + (1 - \lambda_2)f E(y) + b(x, y, \lambda_2)) \\
& = \lambda' f(E(x)) + (1 - \lambda')f(E(y)) + b(x, y, \lambda')
\end{aligned}$$

Therefore $\lambda' \in S$. The claim follows by the fact that $\lambda_1 < \lambda' < \lambda_2$

To complete the proof of Theorem 2.6, let us suppose for the sake of contradiction that

$$\Lambda := [0, 1] \setminus S \neq \emptyset$$

Note that Λ is open since it can be written as $(0, 1) \cap S^c$ and S^c is open by Claim 1.

For fix $\lambda \in \Lambda$. Hence, there is a maximal open interval (λ_1, λ_2) containing λ and contained in Λ . Then $\lambda_1, \lambda_2 \in S$ while $(\lambda_1, \lambda_2) \cap S = \emptyset$, contradicting Claim 2.

This shows that f is necessary sub-b-E-convex.

Conclusion

In this paper we introduced sub-b-E-convex functions. According to definition, it is observed that sub-b-E-convex function can be reduced into E-convex on the condition $b(x, y, \lambda) \leq 0$.

Therefore, the sub-b-E-convex function is a generalization of sub-b-convex function and E-convex function.

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