

On Solving Quadratic Diophantine Equations of Euler Type

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Abstract: The great Swiss mathematician Leonhard Euler in an unpublished proof established that the Quadratic Diophantine Equation of the form $7x^2 + y^2 = 2^n$ always possess solutions in positive integers x and y whenever $n \geq 3$. In this paper, we have considered the more general quadratic Diophantine equation of the form $Dx^2 + y^2 = 2^n$ for particular values of D and derive the complete solutions in non – negative or positive integers for such equations. It turns out that the solutions are based upon the quadratic fields of particular types. Three special cases were discussed to brief the solutions obtained.

Keywords: Diophantine Equations, Polar Form, Euler's Formula, Heegner Numbers.

Introduction

Leonhard Euler is considered to be the greatest contributor of all times in mathematics. His research span almost all modern branches of mathematics or his ideas have created the origin for new developments like the solution to Konisberg Bridge Problem has led to the development of Topology and Graph Theory. Euler's versatile approach in solving mathematical problems is unique and unparalleled. In this paper, we have introduced the general quadratic Diophantine equations of the form $Dx^2 + y^2 = 2^n$ whenever D is one of the three possible Heegner numbers and try to obtain nice formulas exhibiting all non – negative or positive integer solutions to such equations.

Describing the Diophantine Equations

The main objective of this paper is to solve the equations $Dx^2 + y^2 = 2^n$ (1) where $D = 1, 3, 7$ (2) and n is any positive integer, such that x and y are positive integers. The values of D mentioned in (2) forms special class of numbers called Heegner Numbers, which provides unique factorization in the respective quadratic fields.

To accomplish this we make use of particular complex number and express it in polar form through which we can obtain the solutions immediately. Equations (1) described above are Quadratic Diophantine Equations of Euler's type, since Euler gave the complete solution in positive integers to the equation $7x^2 + y^2 = 2^n$.

Expressing through Euler's Formula

First, let us try to express the complex number $(1+i\sqrt{D})^{n-2}$ where D is one of the nine numbers as mentioned in (2) in polar form.

$$1+i\sqrt{D} = r(\cos \alpha + i \sin \alpha) \Rightarrow r \cos \alpha = 1, r \sin \alpha = \sqrt{D}$$

$$\text{From this, we obtain } r^2 = 1+D \Rightarrow r = \sqrt{1+D}; \tan \alpha = \sqrt{D} \Rightarrow \alpha = \tan^{-1}(\sqrt{D})$$

$$\text{Hence the required polar form is } (1+i\sqrt{D})^{n-2} = (1+D)^{(n-2)/2} e^{i(n-2)\tan^{-1}(\sqrt{D})} \quad (3)$$

Now using Euler's Formula in (3), we obtain

$$(1+i\sqrt{D})^{n-2} = (1+D)^{(n-2)/2} \left[\cos\left((n-2)\tan^{-1}(\sqrt{D})\right) + i \sin\left((n-2)\tan^{-1}(\sqrt{D})\right) \right] \quad (4)$$

Solving the Original Problem

In order to solve the original problem as described in (1), we consider the following:

$$\text{Let } y+i\sqrt{D}x = 2^{n/2} (1+D)^{(2-n)/2} \times (1+i\sqrt{D})^{n-2} \quad (5).$$

$$\text{Then its complex conjugate would be } y-i\sqrt{D}x = 2^{n/2} (1+D)^{(2-n)/2} \times (1-i\sqrt{D})^{n-2} \quad (6)$$

Now multiplying (5) and (6) we obtain

$$\begin{aligned} (y+i\sqrt{D}x) \times (y-i\sqrt{D}x) &= 2^n \times (1+D)^{2-n} \times (1+i\sqrt{D})^{n-2} \times (1-i\sqrt{D})^{n-2} \\ Dx^2 + y^2 &= 2^n \times (1+D)^{2-n} \times (1+D)^{n-2} = 2^n \end{aligned}$$

Thus the real numbers x and y , described through complex numbers as in (5) and (6) satisfies (1). Now using (4) in (5), and equating real and imaginary parts, we obtain

$$\sqrt{D}x = 2^{n/2} (1+D)^{(2-n)/2} \times (1+D)^{(n-2)/2} \times \sin\left((n-2)\tan^{-1}(\sqrt{D})\right) \Rightarrow x = \frac{2^{n/2}}{\sqrt{D}} \sin\left((n-2)\tan^{-1}(\sqrt{D})\right) \quad (7)$$

$$y = 2^{n/2} (1+D)^{(2-n)/2} \times (1+D)^{(n-2)/2} \times \cos\left((n-2)\tan^{-1}(\sqrt{D})\right) \Rightarrow y = 2^{n/2} \cos\left((n-2)\tan^{-1}(\sqrt{D})\right) \quad (8)$$

If we now consider the absolute values of x and y from (7) and (8) then $(|x|, |y|)$ provides all non – negative or positive integer solutions to the Quadratic Diophantine Equation $Dx^2 + y^2 = 2^n$.

Special Cases

In this section, we consider D to be three possible Heegner numbers and provide the list of non – negative or positive integers for such values.

5.1 $D = 1$

If we consider the given quadratic Diophantine equation for $D = 1$, then it would be $x^2 + y^2 = 2^n$ (9). The solutions of (9) in view of (7) and (8) are given by

$$x = 2^{n/2} \left| \sin \left((n-2) \tan^{-1}(\sqrt{1}) \right) \right| = 2^{n/2} \left| \sin \left((n-2) \frac{\pi}{4} \right) \right| \quad (10)$$

$$y = 2^{n/2} \left| \cos \left((n-2) \tan^{-1}(\sqrt{1}) \right) \right| = 2^{n/2} \left| \cos \left((n-2) \frac{\pi}{4} \right) \right| \quad (11)$$

We notice that for $n \geq 1$, we get either non – negative or positive values of x, y such that (9) is satisfied.

If n is even number of the form $4k + 2$, then from (10) and (11) we get $x = 0, y = 2^{n/2}$ such that $x^2 + y^2 = 2^n$ is satisfied.

If n is even number of the form $4k$ then from (10) and (11), we get $x = 2^{n/2}, y = 0$ such that $x^2 + y^2 = 2^n$ is satisfied.

If n is odd say $n = 2k - 1$ then from (10) and (11), we get $x = 2^{(n-1)/2}, y = 2^{(n-1)/2}$ such that $x^2 + y^2 = 2^n$ is satisfied. Hence if $n \geq 1$ then (10) and (11) provides either non – negative or positive integer solutions to $x^2 + y^2 = 2^n$.

5.2 $D = 3$

If we consider the given quadratic Diophantine equation for $D = 3$, then it would be $3x^2 + y^2 = 2^n$ (12). The solutions of (12) in view of (7) and (8) are given by

$$x = 2^{n/2} \left| \sin \left((n-2) \tan^{-1}(\sqrt{3}) \right) \right| = 2^{n/2} \left| \sin \left((n-2) \frac{\pi}{3} \right) \right| \quad (13)$$

$$y = 2^{n/2} \left| \cos \left((n-2) \tan^{-1}(\sqrt{3}) \right) \right| = 2^{n/2} \left| \cos \left((n-2) \frac{\pi}{3} \right) \right| \quad (14)$$

We notice that if n is an odd positive integer, then there is no solution to (12) in non – negative or positive integers since at least one of x or y would be non – integer.

If n is an even number of the form $6k + 2$, then from (13) and (14) we get the solutions as $x = 0, y = 2^{n/2}$. Here observe that x is non – negative and y is positive.

If n is even number of the form $6k$ or $6k + 4$ then from (13) and (14) we get the solutions in positive integers as $x = 2^{(n-2)/2}, y = 2^{(n-2)/2}$.

5.3 $D = 7$

If we consider the given quadratic Diophantine equation for $D = 7$, then it would be $7x^2 + y^2 = 2^n$ (15). The solutions of (15) in view of (7) and (8) are given by

$$x = \frac{2^{n/2}}{\sqrt{7}} \left| \sin \left((n-2) \tan^{-1}(\sqrt{7}) \right) \right| \quad (16)$$

$$y = 2^{n/2} \left| \cos \left((n-2) \tan^{-1}(\sqrt{7}) \right) \right| \quad (17)$$

From (16) and (17), we notice that for $n = 3, 4, 5, 6, 7, 8, 9, 10, \dots$ the solutions in positive integers for (15) are respectively given by (1,1); (1,3); (1,5); (2,6); (1,11); (5,9); (7,13); (3,31); $\dots$ These were the solutions that Euler arrived in his work.

Conclusion

Considering the general quadratic Diophantine equation of the form $Dx^2 + y^2 = 2^n$, we have obtained the solutions x, y given by (7) and (8) in this paper. For doing this, we have used the concept of expressing a complex number in polar form and using the fabulous Euler's formula. The expressions (7) and (8) provides the general solution in real numbers for any positive integer D . But if we consider D to be 1 or 3 or 7 then we get either non – negative or positive integer values of x and y satisfying the respective quadratic Diophantine equations. We have provided the complete form of solutions for the cases when $D = 1, 3, 7$ after deriving the general result. It is observed that if $D = 7$ then for $n \geq 3$ we always get positive integer solutions to the quadratic Diophantine equation $7x^2 + y^2 = 2^n$ as Euler mentioned. Thus in this paper, we have not only solved the problem posed by Euler but had generalized the problem and arrived compact solutions for general values of D .

Moreover, the three values of D namely 1, 3 and 7 are three among nine special classes of numbers called Heegner numbers given by 1, 2, 3, 7, 11, 19, 43, 67, 163. Choosing D to be one among these numbers will guarantee that the factorization of Gaussian integers is unique in the respective quadratic fields.

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